Functions for the relationships between VCZAR models and the model properties

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1 The functions

This document describes 7 functions used in connection with the estimation and application of the VCZAR models presented in Tunnicliffe Wilson et al. (2015). The formulae and methodology used in these functions are presented in that book, and also in the document *State space filtering and smoothing using square root methods* which is included in the technical material of Chapter 2 on the website. The 7 functions are described in the following sections. They are listed here as

`VCZARnatural2predictive.m`, `VCZARpreditive2natural.m`, `VCZARIrf.m`,
`VCZARCov.m`, `VCZARCovVT.m`, `VCZARSpecf.m`, `VCZARSpecg.m`

Also described are the MATLAB scripts `VCZARCovSpecTest.m` and `VCZARMapTest.m` which provide consistency checks for the functions.

2 Functions for mapping between model forms

These are `VCZARnatural2predictive.m` and `VCZARpreditive2natural.m`, for which the script `VCZARMapTest.m` provides a consistency check. These two functions map between two of the forms of the VCZAR model, natural and predictive, presented in Chapter 7 of the book. The usage and help lines of the two functions are

```
[Xi, SigXi, M] = VCZARnatural2predictive(Phi, SigPhi, m, p, kappa)
  maps the parameter matrices and disturbance covariance matrix
  from the natural form of VZAR model to the predictive form
```

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```
[Phi, SigPhi, M] = VCZARpredictive2natural(Xi, SigXi, m, p, kappa)
  maps the parameter matrices and disturbance covariance matrix
  from the predictive form of VZAR model to the natural form
```

The predictive form of the model is given in (7.29) of the book, and more compactly in (7.30) and (7.31). The natural form is given by (7.32), (7.33) and (7.34). The nomenclature and notation between MATLAB functions and book equations is straightforward. For the coefficient arrays `Xi` and `Phi`, $\xi_{ijk} \equiv$ `Xi(i,j,k)` and $\varphi_{ijk} \equiv$ `Phi(i,j,k)`. The equations by which one model may be derived from the other are given following the presentation of the models. The model dimension and order are `m` and `p` and `kappa` $\equiv \kappa$, the model rate parameter.

The variance of the disturbance process $e(t)$ in the predictive model form (7.30) is $V_e \equiv$ `SigXi` and the variance of the disturbance process $\epsilon(t)$ in the natural model form (7.34) is $V_\epsilon \equiv$ `SigPhi`. These two process are related by $\epsilon(t) = Me(t)$ where the matrix $M \equiv$ `M`, the output parameter in both functions.

In the script `VCZARMapTest.m` a natural VCZAR model of dimension 3 and order 4 is defined then mapped to the predictive form and mapped back, thus using both of the functions. The absolute differences between the original and returned model coefficients are less than 1.5×10^{-13}.

3 Functions for VCZAR model covariances and spectra

The two functions for model covariances, `VCZARCov.m` and `VCZARCovVT.m` are very similar in their computations, but return different quantities.

The usage and help lines of `VCZARCovVT.m` are as follows

```
[V,T,subm,dem]=VCZARCovVT(Phi,SigPhi,kappa)
  Generates the state variance and transition matrix T of a natural VCZAR model
  with coefficients Phi, SigPhi and parameter kappa.
  The state covariance matrix at a positive lag r is given by C=expm(r*T)*V.
  The series covariance is then C(subm,subm).
  If the model is not stable the function fails and the state eigenvalues
  are returned in dem
```

As stated in these lines, the given model, with coefficients defined by the function arguments, is of the natural form. It is mapped to the predictive form which is used to construct the state space representation, as derived in (7.46). From this the stationary state variance, returned as `V`, is derived using the functions `Tintspade` or `Tintseig`. The state transition matrix T, is returned as `T`. The series covariance at any prescribed lag can be readily constructed from these as described in the help lines.

The usage and help lines of `VCZARCov.m` are as follows

```
[mcov,lvec,dem]=VCZARCov(Phi,SigPhi,kappa,laglim,lagnum)
  Generates the multivariate covariance function of a natural VCZAR model
```

```
  with coefficients Phi, SigPhi and parameter kappa
  at the number lagnum of distinct lags up to a specified lag limit laglim
  A state formulation is used to form the state covariance function
  and the model covariances extracted
  If the model is not stable the function fails and the state eigenvalues
  are returned in dem
```

This function differs from the previous one only in that the quantities `V` and `T` are derived internally then used to construct and extract the series covariances at the lags, and up to the lag limit, prescribed respectively by the input arguments `lagnum` and `laglim`.

The two functions for model spectra, `VCZARSpecf.m` and `VCZARSpecg.m` are very similar in their computations, but return different quantities. They both take the VCZAR in the natural form for their arguments.

The usage and help lines of `VCZARSpecf.m` are as follows

```
[mspec,fvec,wspec,gvec]=VCZARSpecf(Phi,SigPhi,kappa,flim,fnum)
  forms the multivariate spectrum mspec of a natural VCZAR model
  over an equal spacing of fnum un-warped frequencies fvec up to the
  frequency flim returning also the warped spectrum wspec at the
  corresponding warped frequencies gvec
```

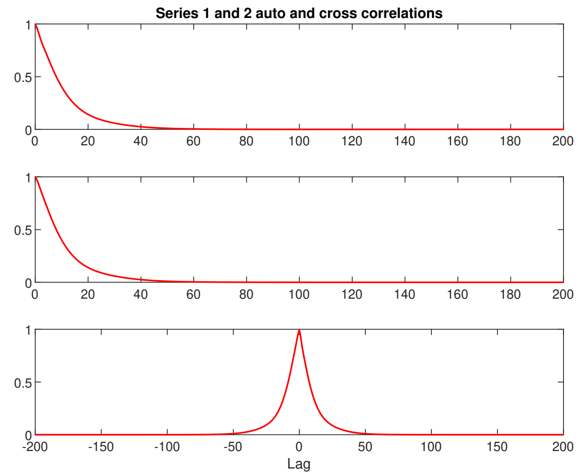
An equally spaced set of frequencies are generated. From these the corresponding warped frequencies are derived and the model warped spectrum constructed as in (7.53) of the book. The unwarped spectrum is derived from this by a Jacobian transformation defined in (7.26).

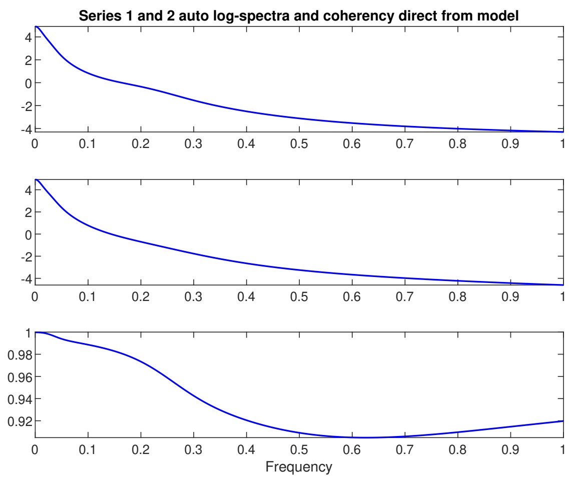
The usage and help lines of `VCZARSpecg.m` are as follows

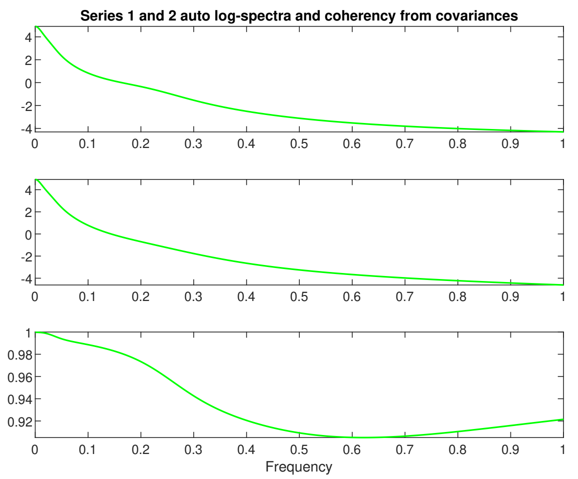
```
[mspec,fvec,wspec,gvec]=VCZARSpecg(Phi,SigPhi,kappa,flim,fnum)
  forms the multivariate spectrum mspec of a natural VCZAR model
  over a range of fnum un-warped frequencies fvec up to the frequency
  flim corresponding to equally spaced frequencies gvec of the warped
  spectrum wspec
```

An equally spaced set of warped frequencies are generated up to a warped frequency limit derived from `flim`. From these the corresponding un-warped frequencies are derived. The model warped spectrum constructed as in (7.53) of the book. The unwarped spectrum is derived from this by a Jacobian transformation defined in (7.26).

The script `VCZARCovSpecTest.m` provides a check on the consistency between the covariance and spectrum calculations. Using `VCZARCov.m`, the covariances are calculated up to a sufficiently high lag for a VCZAR model of dimension 3 and order 4. These are Fourier transformed to the model spectrum and compared with the spectrum calculated directly using `VCZARSpecf.m`. The two results differ because the covariances are necessarily formed at a finite number of discrete lags and the theoretical spectrum is given by the transform over a continuous infinite range of lags. However, the discrepancy can be substantially reduced by an error correction device based on the same principle as Simpson's rule for reducing the numerical quadrature approximation to an integral. Plots of the resulting covariance functions and spectra for two of the series are shown here


Series 1 and 2 auto and cross correlations
Lag


Series 1 and 2 auto log-spectra and coherency direct from model
Frequency


Series 1 and 2 auto log-spectra and coherency from covariances
Frequency

4 Function for the model impulse and step responses

The function `VCZARIrf` forms both the impulse and step response of one series to another for a VCZAR model. At the present time it is restricted to a model of dimension 2. The usage and help lines of the function are as follows

```
[irf,srf,lvec]=VCZARIrf(Phi,SigPhi,kappa,laglim,lagnum,feedback)
  Generates the impulse and step response of an output series to an input series
  for a bivariate natural VCZAR model with coefficients Phi, SigPhi and parameter
  kappa at the number lagnum of distinct lags up to a specified lag limit laglim.
  A state formulation is used with an input of impulse form, and Matlab expm
  function for the transition step. The first series is the input,
  the second the output.
```

The function generates, at the lag times, the output values of the second series given that the first series is identically zero except for a unit impulse at time zero. Past values of the output series are also assumed to be zero. The lines above explain the first 5 function arguments. The argument `feedback`, when set to zero, removes all the feedback terms to the first series in the predictive form of the model, setting $\xi_{1,j,k} = 0$ for all j, k. This generates the open-loop response. Otherwise, the closed loop response is generated.

The results are determined by first transforming the natural model to the predictive form which is then used to construct the state space representation. Values of the initial state vector corresponding to the input are set to those of a unit step. Those corresponding to the output vector are similar but scaled by the contemporaneous dependence of the output on the input deriving from the regression of the disturbance of output 2 on that of output 1. The noise-free step response is then simply derived by application of the finite step transition matrix to the initial state. This is differentiated numerically to give the impulse response. The returned quantities `irf`, `srf` and `lvec` hold respectively the impulse and step responses and their lags.

References

G. Tunnicliffe Wilson, M. Reale, and J. Haywood. *Models for dependent time series.* New York, CRC Press, 2015.