# Calculation of the covariance function of a vector autoregression

Granville Tunnicliffe Wilson*

May 1, 2017

## 1 From the Yule-Walker equations

Let $\Gamma_k$ be the covariance at lag $k$ of a VAR($p$) model for an $m$-dimensional process $x_t$, with given coefficients $\Phi_k$ and innovation variance $V$:

$$x_t = \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + \cdots + \Phi_p x_{t-p} + e_t, \tag{1}$$

The Yule-Walker equations given in the book as (2.5) and (2.6) can be combined into a $(p+1) \times (p+1)$ block matrix equation:

$$\begin{pmatrix} 1 & -\Phi_1 & -\Phi_2 & \cdots & -\Phi_p \end{pmatrix} \begin{pmatrix} \Gamma_0 & \Gamma_1 & \cdots & \cdots & \Gamma_{(p)} \\ \Gamma_{-1} & \Gamma_0 & \cdots & \cdots & \Gamma_{(p-1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \Gamma_{-p} & \Gamma_{-(p-1)} & \cdots & \cdots & \Gamma_0 \end{pmatrix} = \begin{pmatrix} V & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{2}$$

Now let $\Phi_0 = I$ and write $\Phi$ for $-Phi$, reversing the sign convention in (1). Then (2)can be re-arranged as linear equations for the elements of $\Gamma_0 \ldots \Gamma_p$. However, they cannot be solved by block matrix algebra such as that used in the Whittle recursive method to derive the coefficients $\Phi_k$ from the covariances $\Gamma_k$. In terms of the elements of the quantities involved (and recalling $\Gamma_{-k} = \Gamma'_k$) we can write the set of equations:

$$\sum_{k_2=0}^{p-k_1} \sum_{j2=1}^{m} \Phi_{i_1\, j_2\, k_1+k_2} \Gamma_{j_1\, j_2\, k_2} + \sum_{k_2=1}^{k_1} \sum_{j2=1}^{m} \Phi_{i_1\, j_2\, k_1-k_2} \Gamma_{j_2\, j_1\, k_2} \quad = \quad V_{i_1\, j_1} \delta_{k_1,0} \tag{3}$$

$$\text{for} \quad i_1 = 1 \ldots m, j_1 = 1 \ldots m, k_1 = 0 \ldots p,$$

where $\delta_{i,j} = 1$ if $i = j$, else is zero. The coefficient of $\Gamma_{i_2\, j_2\, k_2}$ in the equations is then

$$\Phi_{i_1\, j_2\, k_1+k_2} \delta_{j_1,i_2} + \Phi_{i_1\, i_2\, k_1-k_2} \delta_{j_1,j2}.$$

*Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, UK.

These are the equations solved by the MATLAB function `VARcovfunX.m` using indices for the rows and columns which are incremented as the variables $i_1, j_1, k_1$ and $i_2, j_2, k_2$ loop through their ranges. Because $\Gamma_0$ is symmetric, the rows and columns of its sub-diagonal elements are removed from the equations and its semi-diagonals used as unknowns.

After solving for $\Gamma_0 \dots \Gamma_p$ the higher lag covariances are calculated using (2.25) in the book.

# 2 From the covariance generating function

Section 2.10 of the book presents the covariance generating function of a VAR($p$) model which may be expressed in terms of its positive and negative powers of $z$ as

$$\Gamma(z) = \Phi(z)^{-1} V \left\{\Phi(z^{-1})'\right\}^{-1} = \Phi(z)^{-1} A(z) + A(z^{-1})' \left\{\Phi(z^{-1})'\right\}^{-1}, \tag{4}$$

where

$$A(z) = A_0 + A_1 z + \dots A_p z^p, \tag{5}$$

with $A_0$ constrained to be upper triangular for uniqueness. Then on expanding we obtain

$$\Phi(z)^{-1} A(z) = A_0 + \Gamma_1 z + \Gamma_2 z^2 + \dots \tag{6}$$

with $\operatorname{Var} x_t = \Gamma_0 = A_0 + A_0'$.

The equations to be solved for $A(z)$ come from multiplying out (4) as

$$V = A(z)\Phi(z^{-1})' + \Phi(z)A(z^{-1})'. \tag{7}$$

Equating coefficients of powers of $z$ gives linear equations for the coefficients of $A(z)$ as

$$\sum_{k_2=k_1}^{p} \sum_{j2=1}^{m} A_{i_1\, j_2\, k_2} \Phi_{j_1\, j_2\, k_2-k_1} + \sum_{k_2=0}^{p-k_1} \sum_{j2=1}^{m} \Phi_{i_1\, j_2\, k_1+k_2} A_{j_1\, j_2\, k_2} \quad = \quad V_{i_1\, j_1} \delta_{k_1,0} \tag{8}$$

$$\text{for} \quad i_1 = 1 \dots m, j_1 = 1 \dots m, k_1 = 0 \dots p.$$

The coefficient of $A_{i_2\, j_2\, k_2}$ in the equations is then

$$\Phi_{j_1\, j_2\, k_2 - k1} \delta_{i_1, i_2} + \Phi_{i_1\, j_2\, k_2+k_1} \delta_{j_1, i2}.$$

We are here using the same sign convention for $\Phi_k$ as in (3). These are the equations solved by the MATLAB function `VARcovfun.m` using indices for the rows and columns which are incremented as the variables $i_1, j_1, k_1$ and $i_2, j_2, k_2$ loop through their ranges. They are similar, but not identical, to the equations (3). After obtaining $A(z)$ the covariances are generated by expanding (6) using the recurrence

$$H_k = A_k + \sum_{i=1}^{p} \Phi_i H_{k-i}; \quad k = 1, 2, \dots \tag{9}$$

taking $H_0 = A_0$ and $H_k = 0$ for $k < 0$, and with the original sign convention for $\Phi_k$ restored. Then $\Gamma_0 = H_0 + H_0'$ and $\Gamma_k = H_k$ for $k > 0$.