A formula for the inverse covariance matrix of a vector autoregression

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1 The formula

Let Γ_k be the covariance at lag k of a VAR(p) model for an m-dimensional process x_t, with given coefficients Φ_k and innovation variance V_e:

$$x_t + \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + \cdots + \Phi_p x_{t-p} = e_t, \tag{1}$$

having the reversed time or backward representation:

$$x_t + \tilde{\Phi}_1 x_{t+1} + \tilde{\Phi}_2 x_{t+2} + \cdots + \tilde{\Phi}_p x_{t+p} = \tilde{e}_t. \tag{2}$$

Note that the coefficients Φ_k and $\tilde{\Phi}_k$ here have the reverse of the sign used conventionally in the book. This is notationally convenient for the derivation.

Let the variance matrix of the stacked vector X_n of the finite span of series values x_1, ..., x_n be:

$$\text{Var} X_n = \text{Var}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \Gamma_0 & \Gamma_{-1} & \cdots & \Gamma_{-(n-1)} \\ \Gamma_1 & \Gamma_0 & \cdots & \Gamma_{-(n-2)} \\ \vdots & \ddots & \ddots & \vdots \\ \Gamma_{(n-1)} & \Gamma_{(n-2)} & \cdots & \Gamma_0 \end{pmatrix} = \tilde{G}_n. \tag{3}$$

Then the book formula (2.61) for the inverse covariance matrix is

$$\tilde{G}_n^{-1} = M' W_n M - N' \tilde{W}_p N \tag{4}$$

where the $n \times n$ block matrix M and $p \times n$ block matrix N are

$$M = \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0 & 0 \\ \Phi_1 & I & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \Phi_p & \cdots & \Phi_1 & I & 0 & \cdots & 0 \\ 0 & \Phi_p & \cdots & \Phi_1 & I & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \Phi_p & \cdots & \Phi_1 & I \end{pmatrix} \tag{5}$$

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and

$$N = \begin{pmatrix} \tilde{\Phi}_p & 0 & 0 & \cdots & 0 & 0 & 0 \\ \tilde{\Phi}_{p-1} & \tilde{\Phi}_p & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \tilde{\Phi}_1 & \cdots & \tilde{\Phi}_{p-1} & \tilde{\Phi}_p & 0 & \cdots & 0 \end{pmatrix} \tag{6}$$

and the matrices W_n and $\tilde{W}_p$ in (4) are respectively $n \times n$ and $p \times p$ block diagonal with entries V_e^{-1} and $\tilde{V}_e^{-1}$ along their respective diagonals.

2 Derivation of the formula

Let the $n \times n$ block matrix A be given by

$$A = \left(\begin{array}{cc} I_p & 0_{p\times(n-p)} \\ \hline \multicolumn{2}{c}{B} \end{array} \right), \tag{7}$$

where, to clarify, I_p indicates a diagonal matrix of p identity matrices of size m, and similarly for $0_{p\times(n-p)}$. The $(n-p) \times n$ block matrix B is given by

$$B = \begin{pmatrix} \Phi_p & \Phi_{p-1} & \cdots & \Phi_1 & I & 0 & \cdots & 0 \\ 0 & \Phi_p & \Phi_{p-1} & \cdots & \Phi_1 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Phi_p & \Phi_{p-1} & \cdots & \Phi_1 & I \end{pmatrix}. \tag{8}$$

Note that the upper left and lower right $p \times p$ block components of B are respectively

$$C = \begin{pmatrix} \Phi_p & \Phi_{p-1} & \cdots & \Phi_1 \\ 0 & \Phi_p & \cdots & \Phi_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \Phi_p \end{pmatrix}, \qquad \text{and} \qquad D = \begin{pmatrix} I & 0 & \cdots & 0 \\ \Phi_1 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \Phi_{p-1} & \cdots & \Phi_1 & I \end{pmatrix}. \tag{9}$$

Then from (1), application of A to X_n leaves the first p terms unchanged and transforms the remainder to their innovations:

$$AX_n = \left(\begin{array}{c} x_1 \\ \vdots \\ x_p \\ \hline e_{p+1} \\ e_{p+2} \\ \vdots \\ e_n \end{array} \right), \tag{10}$$

so that on taking variances

$$A\tilde{G}_n A' = \left(\begin{array}{c|c} \tilde{G}_p & 0 \\ \hline 0 & V_{(n-p)} \end{array} \right), \tag{11}$$

where V_{n-p} consists of $n-p$ block diagonals all of V_e. Consequently

$$\tilde{G}_n^{-1} = A' \left(\begin{array}{c|c} \tilde{G}_p^{-1} & 0 \\ \hline 0 & W_{(n-p)} \end{array} \right) A, \tag{12}$$

where $W_{n-p} = V_{n-p}^{-1}$.

Similarly define, in terms of the coefficients of the backward model:

$$\tilde{A} = \left(\begin{array}{cc} \multicolumn{2}{c}{\tilde{B}} \\ \hline 0_{p\times(n-p)} & I_p \end{array} \right), \tag{13}$$

where

$$\tilde{B} = \begin{pmatrix} I & \tilde{\Phi}_1 & \cdots & \tilde{\Phi}_{p-1} & \tilde{\Phi}_p & 0 & \cdots & 0 \\ 0 & I & \tilde{\Phi}_1 & \cdots & \tilde{\Phi}_{p-1} & \tilde{\Phi}_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I & \tilde{\Phi}_1 & \cdots & \tilde{\Phi}_{p-1} & \tilde{\Phi}_p \end{pmatrix} \tag{14}$$

with upper left and lower right $p \times p$ block components respectively

$$\tilde{C} = \begin{pmatrix} I & \tilde{\Phi}_1 & \cdots & \tilde{\Phi}_{p-1} \\ 0 & I & \cdots & \tilde{\Phi}_{p-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I \end{pmatrix}, \qquad \text{and} \qquad \tilde{D} = \begin{pmatrix} \tilde{\Phi}_p & 0 & \cdots & 0 \\ \tilde{\Phi}_{p-1} & \tilde{\Phi}_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{\Phi}_1 & \cdots & \tilde{\Phi}_{p-1} & \tilde{\Phi}_p \end{pmatrix}. \tag{15}$$

Then

$$\tilde{A}X_n = \left(\begin{array}{c} \tilde{e}_1 \\ \tilde{e}_2 \\ \vdots \\ \tilde{e}_{n-p} \\ \hline x_{n-p+1} \\ \vdots \\ x_n \end{array} \right), \tag{16}$$

and on taking variances

$$\tilde{A}\tilde{G}_n\tilde{A}' = \left(\begin{array}{c|c} \tilde{V}_{(n-p)} & 0 \\ \hline 0 & \tilde{G}_p \end{array} \right), \tag{17}$$

where $\tilde{V}_{n-p}$ consists of $n-p$ block diagonals all of $V_{\tilde{e}}$. Consequently

$$\tilde{G}_n^{-1} = \tilde{A}' \left(\begin{array}{c|c} \tilde{W}_{(n-p)} & 0 \\ \hline 0 & \tilde{G}_p^{-1} \end{array} \right) \tilde{A}, \tag{18}$$

where $\tilde{W}_{n-p} = \tilde{V}_{n-p}^{-1}$.

Now equate the lower right $p \times p$ block matrices of (12) and (18) which gives

$$D'W_pD = \tilde{D}'\tilde{W}_p\tilde{D} + \tilde{G}_p^{-1} \tag{19}$$

and hence

$$\tilde{G}_p^{-1} = D'W_pD - \tilde{D}'\tilde{W}_p\tilde{D}. \tag{20}$$

Note that by equating the upper left $p \times p$ block matrices of (12) and (18) gives also

$$\tilde{G}_p^{-1} + C'W_pC = \tilde{C}'\tilde{W}_p\tilde{C} \tag{21}$$

and hence

$$\tilde{G}_p^{-1} = \tilde{C}'\tilde{W}_p\tilde{C} - C'W_pC. \tag{22}$$

However, it is from (20) that we substitute for $\tilde{G}_p^{-1}$ into (12), which may be expressed using (7) as

$$\left(\begin{array}{c|c} G_p^{-1} & 0 \\ \hline 0 & 0 \end{array}\right) + B'W_{(n-p)}B = \left(\begin{array}{c|c} D'W_pD & 0 \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} \tilde{D}'\tilde{W}_p\tilde{D} & 0 \\ \hline 0 & 0 \end{array}\right) + B'W_{(n-p)}B. \tag{23}$$

We now note that

$$M = \left(\begin{array}{cc} D & 0 \\ \hline \multicolumn{2}{c}{B} \end{array}\right) \tag{24}$$

so that

$$M'W_nM = \begin{pmatrix} D' \\ 0 \end{pmatrix} W_p \begin{pmatrix} D & 0 \end{pmatrix} + B'W_{(n-p)}B. \tag{25}$$

Similarly using

$$N = \begin{pmatrix} \tilde{D} & 0 \end{pmatrix} \tag{26}$$

we obtain

$$N'\tilde{W}_nN = \begin{pmatrix} \tilde{D}' \\ 0 \end{pmatrix} \tilde{W}_p \begin{pmatrix} \tilde{D} & 0 \end{pmatrix}. \tag{27}$$

Combining (25) and (27) as in (4) we see that they agree with (23).

For the expression (2.61) in the book reverse the signs of M and N, which does not affect (4), and set $\Phi_0 = \tilde{\Phi}_0 = -I$.