The structure of the error term in the VZAR model

Granville Tunnicliffe Wilson*

May 15, 2017

1 The different model forms

In Section 6.3 of Tunnicliffe Wilson et al. (2015) three different forms are given of what is termed the VZAR model.

The first of these is called the *predictive* form of model:

$$\begin{aligned} x_t &= \xi_1 x_{t-1} + \xi_2 Z x_{t-1} + \cdots + \xi_p Z^{p-1} x_{t-1} + e_t && (1) \\ &= \xi(Z) B x_t + e_t, && (2) \end{aligned}$$

where

$$\xi(Z) = \xi_1 + \xi_2 Z + \cdots + \xi_p Z^{p-1} \quad (3)$$

and e_t is the white noise linear innovation series with

$$\text{Var}(e_t) \;=\; V_e. \quad (4)$$

The second is called the *natural* form of model:

$$x_t = \varphi_1 Z x_t + \varphi_2 Z^2 x_t + \cdots + \varphi_p Z^p x_t + n_t \quad (5)$$

or

$$\varphi(Z) x_t = n_t, \quad (6)$$

with

$$\varphi(Z) = I - \varphi_1 Z - \varphi_2 Z^2 - \cdots - \varphi_p Z^p \quad (7)$$

and disturbance term n_t following a VAR(1) process with scalar coefficient θ:

$$n_t = \theta n_{t-1} + \varepsilon_t, \quad (8)$$

where ε_t is multivariate white noise with variance V_ε related to the variance V_n of n_t by $V_\varepsilon = (1 - \theta^2) V_n$.

*Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, UK.

The third is the *general* form of model:

$$\begin{aligned} x_t &= Z_\rho(\zeta_1 x_t + \zeta_2 Z x_t + \cdots + \zeta_p Z^{p-1} x_t) + n_t \\ &= Z_\rho \zeta(Z) x_t + n_t \end{aligned} \tag{9}$$

where

$$\zeta(Z) = \zeta_1 + \zeta_2 Z + \cdots + \zeta_p Z^{p-1}. \tag{10}$$

The disturbance term term n_t in (9) now follows a standard VAR(1) process with *scalar* autoregressive coefficient ρ:

$$n_t = \rho n_{t-1} + \varepsilon_t, \tag{11}$$

where ε_t is multivariate white noise with variance V_ε related to the variance V_n of n_t by $V_\varepsilon = (1-\rho^2)V_n$.

The general form encompasses the predictive and natural forms of model by setting ρ respectively to 0 and θ, so we will present the development of just this general form. Its motivation comes from the equation for defining a minimum variance linear prediction of the weighted function of future values given by:

$$\mathrm{L}\left(Z_\rho^{-1} x_t \,\middle|\, x_\mathrm{P}\right) = \zeta_1 x_t + \zeta_2 Z x_t + \cdots + \zeta_p Z^{p-1} x_t, \tag{12}$$

where the ζ_k are matrix coefficients.

The assumption that this predictor is minimum variance among all linear functions of all present and past variables $x_\mathrm{P} = \{x_t, x_{t-1}, \ldots\}$, requires that the prediction error

$$\tilde{n}_t = Z_\rho^{-1} x_t - \mathrm{L}\left(Z_\rho^{-1} x_t \,\middle|\, x_\mathrm{P}\right) \tag{13}$$

is uncorrelated with all elements of x_P.

We now use the simple lemma that for any pair of stationary series α_t and β_t with cross-covariances

$$\mathrm{Cov}\left\{\alpha_t, \beta_{t-k}\right\} = \Gamma_k \tag{14}$$

and operator $A(B) = \sum_k A_k B^k$, we have

$$\mathrm{Cov}\left\{A(B)\alpha_t, \beta_t\right\} = \mathrm{Cov}\left\{\alpha_t, A(B^{-1})\beta_t\right\}, \tag{15}$$

because each expression is verified as being equal to $\sum_k A_k \Gamma_{-k}$. We also note that if $A(B) = Z_\rho^{-1}$, then $A(B^{-1}) = Z_\rho$.

So from (10), (12) and (13) we have that for $k = 0, 1, 2, \ldots$

$$\begin{aligned} 0 = \mathrm{Cov}(\tilde{n}_t, x_{t-k}) &= \mathrm{Cov}\left\{Z_\rho^{-1} x_t - \zeta(Z) x_t, x_{t-k}\right\} \\ &= \mathrm{Cov}(Z_\rho^{-1} n_t, x_{t-k}) = \mathrm{Cov}(n_t, Z_\rho\, x_{t-k}), \end{aligned} \tag{16}$$

where we have introduced n_t from the model equation (9). It remains to show that n_t has the VAR(1) structure given in (9). To do this we write

$$Z_\rho = \frac{B-\rho}{1-\rho B} = (1-\rho B^{-1}) \frac{B}{1-\rho B} \tag{17}$$

so that, for $k = 0, 1, 2, \ldots$

$$\begin{aligned}
\operatorname{Cov}(n_t, Z_\rho\, x_{t-k}) &= \operatorname{Cov}\left\{n_t, (1-\rho B^{-1})\frac{B}{1-\rho B}x_{t-k}\right\} &(18)\\
&= \operatorname{Cov}\left\{(1-\rho B)n_t, \frac{B}{1-\rho B}x_{t-k}\right\} = 0, &(19)
\end{aligned}$$

where we have once again applied the simple lemma. Then letting $(1-\rho B)n_t = \varepsilon_t$, we see that $\operatorname{Cov}\{\varepsilon_t, (1-\rho B)^{-1}x_{t-k}\} = 0$ for $k = 1, 2, \ldots$. Now the linear space of variables $(1-\rho B)^{-1}x_{t-k}$ for $k = 1, 2, \ldots$ is identical to the linear space of variables x_{t-k} for $k = 1, 2, \ldots$. Also ε_t lies in the space of x_{t-k} for $k = 0, 1, 2, \ldots$ so must be proportional to the linear innovation in x_t and therefore white noise.

References

G. Tunnicliffe Wilson, M. Reale, and J. Haywood. *Models for dependent time series.* New York, CRC Press, 2015.